

# THE MAPS OF MATRICES AND PORTRAIT MAPS OF DENSITY OPERATORS OF COMPOSITE AND NONCOMPOSITE SYSTEMS

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## Abstract

We obtain a new inequality for arbitrary Hermitian matrices. We describe particular linear maps called the matrix portrait of arbitrary  $N \times N$  matrices. The maps are obtained as analogs of partial tracing of density matrices of multipartite qudit systems. The structure of the maps is inspired by “portrait” map of the probability vectors corresponding to the action on the vectors by stochastic matrices containing either unity or zero matrix elements. We obtain new entropic inequalities for arbitrary qudit states including a single qudit and an discuss entangled single qudit state. We consider in detail the examples of  $N = 3$  and 4. Also we point out a possible use of entangled states of systems without subsystems (e.g., a single qudit) as a resource for quantum computations.

**Keywords:** quantum channels, positive maps, partial trace, quantum entanglement, nonseparable states.

## 1 Introduction

The quantum states of arbitrary systems including the spin (qudit) systems are identified with the density matrices  $\rho$  [1–3], which are Hermitian nonnegative matrices  $\rho = \rho^\dagger$  and  $\rho \geq 0$  with  $\text{Tr } \rho = 1$ . The influence of different devices and measurements on the system state is associated with a map of the density matrix  $\rho \rightarrow \rho' = \Phi(\rho)$ , where the matrix  $\rho'$  belongs to the set of density matrices. If the map is a linear transform of the density matrix, it is called the positive map. Mathematical and physical aspects of the positive maps have been discussed in [4, 5].

Particular positive maps corresponding to the transformation of the density matrix of a physical system inspired by the interaction of the system with environment are called the completely positive maps. A set of completely positive maps is the subset of positive maps. In quantum information theory, the completely positive maps are called the quantum channels [6]. If the function  $\Phi(\rho)$  does not describe the linear transform of the density matrix, the map is called the nonlinear map or the nonlinear quantum channel [7]. In the probability representation of quantum states [8–12], the examples of nonlinear positive maps were given in [13] on the example of unitary tomograms of qudit states.

In composite quantum systems, the entanglement phenomenon [14] corresponding to strong quantum correlations between the subsystems takes place. In some cases, the entanglement can be detected applying the portrait map of the density matrix of the composite-system state [15, 16]. The portrait of the density matrix is a very particular example of the positive maps of the density matrix. The common properties of the probability vectors within the classical and quantum frameworks were studied in [17]. Recently, it was observed that the properties of quantum entanglement and other aspects of quantum correlations [7, 18–22] existing in composite quantum systems, e.g., in the form of entropic inequalities [20, 23–27], exist also in a single qudit system.

In fact, the mathematical structure for formulating the quantum correlation properties of composite systems in the form of equalities and inequalities for the density matrices of such systems can be found for the density matrix of the systems without subsystems as well. This fact provides the possibility to obtain the entanglement properties and new entropic inequalities for density matrices of the systems without subsystems. One can also formulate all Bell-like inequalities [28–30] and study the violation of these inequalities for the systems without subsystems.

It is assumed that the resource for developing fast quantum computations is associated with the properties of entangled states of composite quantum systems [31]. For example, the system of  $N$  qubits is such a system. From the observation above mentioned follows that an analogous resource can be associated with entangled states of a single qudit with large spin  $j$ , such that  $2j + 1 = 2^N$ . So, for  $N = 2$  this means that the system of two qubits, i.e., two spins with  $j = 1/2$ , has the same entanglement resource as one qudit with  $j = 3/2$ .

The aim of our work is to show that any  $N \times N$  density matrix  $\rho$  of a quantum system satisfies the same entropic inequalities associated with the properties of different matrices  $\Phi_1(\rho), \Phi_2(\rho), \dots, \Phi_n(\rho)$ , either being identified or independently of the identification of the matrices  $\Phi_k(\rho)$  with the density matrices of the subsystem states of the system under consideration. From mathematical point of view, our goal is to present an analog of the entropic inequality for an arbitrary Hermitian matrix that seems to be a new inequality in matrix theory.

This new inequality, being applied to nonnegative trace-class Hermitian matrices, provides a new entropic inequality for quantum-state density matrices. The structure of the inequality makes clear the possibility to introduce the notion of entanglement and other quantum correlation aspects as characteristics of both composite and noncomposite quantum systems. This idea is coherent with the approach to hidden variables for spin  $j = 1$  state [32]. Since the entanglement phenomenon corresponding to quantum correlations of subsystems of composite systems provides a resource for quantum computations, we also consider the possibility to use the entangled states of noncomposite systems as such a resource as well.

This paper is organized as follows.

In Sec. 2, we give the explanation of new inequalities valid for arbitrary Hermitian matrices. In Sec. 3, we consider a few examples of matrix inequalities for three-dimensional and four-dimensional matrices. In Sec. 4, we present our conclusions and the prospectives.

## 2 Linear Map of $N \times N$ Matrices

The  $N \times N$  matrix  $A$  with matrix elements  $A_{jk}$  can be mapped onto the  $N^2$ -vector  $\vec{A}$ . For example,

for  $N = 2$ ,  $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ , and we define the vector  $\vec{A}$  as the column vector  $\vec{A} = \begin{pmatrix} A_{11} \\ A_{12} \\ A_{21} \\ A_{22} \end{pmatrix}$ ; the

map is invertible.

For the map of indices  $11 \leftrightarrow 1$ ,  $12 \leftrightarrow 2$ ,  $21 \leftrightarrow 3$ , and  $22 \leftrightarrow 4$ , the vector  $\vec{A}$  has four components  $\vec{A} = (A_1, A_2, A_3, A_4)$ . We can define an analogous map for an arbitrary  $N$ .

The linear map of matrices  $A \rightarrow A'$  can be considered as a map of vectors, i.e.,  $A_{jk} \rightarrow A'_{jk} = \sum_{m,n=1}^N B_{jk,mn} A_{mn}$  can be considered as the relation  $A_\alpha \rightarrow A'_\alpha = \sum_{\beta=1}^{N^2} b_{\alpha,\beta} A_\beta$ .

For nonnegative Hermitian matrices  $A$ , such that  $\text{Tr } A = 1$  and the eigenvalues of the matrix are nonnegative, the linear map  $A \rightarrow A'$  is called the positive map, if the matrices  $A'$  have the same properties. The structure of matrices  $b_{\alpha,\beta}$  and  $B_{jk,mn}$  has been studied in [5].

In this paper, we consider specific linear maps of arbitrary complex matrices  $A$ .

Let  $N = nm$  with  $n$  and  $m$ , the integers. There exist two matrices  $A_1$  and  $A_2$  obtained by applying the linear maps,  $A \rightarrow A_1$  and  $A \rightarrow A_2$ . We define the maps as follows.

Let the matrix  $A$  be presented in the block form

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}, \quad (1)$$

where the blocks  $a_{jk}$  ( $j, k = 1, 2, \dots, n$ ) are the  $m \times m$  matrices. The maps we defined read

$$A \rightarrow A_1 = \begin{pmatrix} \text{Tr } a_{11} & \text{Tr } a_{12} & \cdots & \text{Tr } a_{1n} \\ \text{Tr } a_{21} & \text{Tr } a_{22} & \cdots & \text{Tr } a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \text{Tr } a_{n1} & \text{Tr } a_{n2} & \cdots & \text{Tr } a_{nn} \end{pmatrix}, \quad A \rightarrow A_2 = \sum_{k=1}^n a_{kk} = \sum_{k,j=1}^n a_{kj} \delta_{kj}. \quad (2)$$

Thus, we obtained two matrices: the  $n \times n$  matrix  $A_1$  and the  $m \times m$  matrix  $A_2$ . The constructed map preserves the trace, i.e.,  $\text{Tr } A = \text{Tr } A_1 = \text{Tr } A_2$ , and if  $A^\dagger = A$ , then  $A_1^\dagger = A_1$  and  $A_2^\dagger = A_2$ . The map  $A \rightarrow A_1 \otimes A_2$  is the nonlinear map.

The map constructed has the invariance properties. If we replace the matrix  $A$  by the matrix

$$A_u = (1_n \otimes u_m) A (1_n \otimes u_m^\dagger),$$

where  $u_m$  is the unitary  $m \times m$  matrix, and  $1_n$  is the  $n \times n$  identity matrix, the matrix  $A_{1u}$  (obtained by the described procedure from the matrix  $A_u$ ) does not depend on the unitary matrix  $u_n$ , i.e.,  $A_{1u} = A_1$ .

Analogously, if the matrix  $A$  is replaced by the matrix

$$\tilde{A}_u = (u_n \otimes 1_m) A (u_n^\dagger \otimes 1_m),$$

one has the property  $A_{2u} = A_2$ . We can prove that for an arbitrary Hermitian  $N \times N$  matrix  $A = A^\dagger$ , such that  $\text{Tr } A = 1$  and  $A \geq 0$ , the inequality  $-\text{Tr } A \ln A \leq -\text{Tr } A_1 \ln A_1 - \text{Tr } A_2 \ln A_2$  is valid.

If  $\text{Tr } A^2 = \text{Tr } A = 1$  (pure state), the set of eigenvalues of  $A_1$  and  $A_2$  is the same set. It is true for an arbitrary possible factorization  $N = nm = n'm'$ . With the matrix  $A$ , one can associate an analog of the mutual information with respect to the decomposition  $N = nm$ . We introduce the mutual matrix information

$$I_{nm} = -\text{Tr } A_1 \ln A_1 - \text{Tr } A_2 \ln A_2 + \text{Tr } A \ln A. \quad (3)$$

If we introduce  $N \times N$  matrices  $\tilde{A}_1$  and  $\tilde{A}_2$ ,

$$\tilde{A}_1 = \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \tilde{A}_2 = \begin{pmatrix} A_2 & 0 \\ 0 & 0 \end{pmatrix}, \quad (4)$$

the mutual information is determined as

$$I_{nm} = \text{Tr} \left( A \ln A - \tilde{A}_1 \ln \tilde{A}_1 - \tilde{A}_2 \ln \tilde{A}_2 \right) \geq 0. \quad (5)$$

Now we are in a position to formulate a new statement on the properties of matrices.

Given  $N \times N$  matrix  $A$  [Eq. (1)] such that  $A = A^\dagger$ ,  $A \geq 0$ , and  $\text{Tr } A = 1$ , this matrix is presented in the block form with  $n^2$  block  $m \times m$  matrices  $a_{jk}$  having matrix elements  $(a_{jk})_{\alpha\beta}$   $\alpha, \beta = 1, 2, \dots, m$ , and  $N = nm$ , then the inequality holds

$$\begin{aligned} & -\text{Tr} \left\{ \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \ln \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \right\} \\ & \leq -\text{Tr} \left\{ \begin{pmatrix} \text{Tr } a_{11} & \text{Tr } a_{12} & \cdots & \text{Tr } a_{1n} \\ \text{Tr } a_{21} & \text{Tr } a_{22} & \cdots & \text{Tr } a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \text{Tr } a_{n1} & \text{Tr } a_{n2} & \cdots & \text{Tr } a_{nn} \end{pmatrix} \ln \begin{pmatrix} \text{Tr } a_{11} & \text{Tr } a_{12} & \cdots & \text{Tr } a_{1n} \\ \text{Tr } a_{21} & \text{Tr } a_{22} & \cdots & \text{Tr } a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \text{Tr } a_{n1} & \text{Tr } a_{n2} & \cdots & \text{Tr } a_{nn} \end{pmatrix} \right\} \\ & - \text{Tr} \{ (a_{11} + a_{22} + \cdots + a_{nn}) \ln (a_{11} + a_{22} + \cdots + a_{nn}) \}. \end{aligned} \quad (6)$$

At  $N \neq nm$ , we choose an integer  $s$ , such that the number  $\tilde{N} = N + s = nm$ , and construct the  $\tilde{N} \times \tilde{N}$  matrix

$$\tilde{A} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}. \quad (7)$$

After presenting the matrix  $\tilde{A}$  in an analogous block form and taking into account zero matrix elements, we arrive at the inequality

$$-\text{Tr} (\tilde{A} \ln \tilde{A}) = -\text{Tr} (A \ln A) \leq -\text{Tr} (A_1 \ln A_1) - \text{Tr} (A_2 \ln A_2). \quad (8)$$

At  $\text{Tr } A = \mu$  and  $A \geq 0$ , we obtain an analogous inequality, where the term  $\mu \ln \mu$  is taken into account; the inequality reads

$$-\text{Tr} (A \ln A) \leq -\text{Tr} (A_1 \ln A_1) - \text{Tr} (A_2 \ln A_2) + \text{Tr } A \ln (\text{Tr } A). \quad (9)$$

If the  $N \times N$  matrix  $A$  and the  $\tilde{N} \times \tilde{N}$  matrix  $\tilde{A}$  with  $\text{Tr } A = \text{Tr } \tilde{A} = \mu$ , have the minimum eigenvalue  $A_0$ , which can be either positive or negative, then for an arbitrary positive number  $x \geq |A_0|$  the matrix  $A'(x) = \tilde{A} + x1_{\tilde{N}} \geq 0$ , and for this matrix we obtain the inequality

$$-\text{Tr} (A'(x) \ln A'(x)) \leq -\text{Tr} (A'_1(x) \ln A'_1(x)) - \text{Tr} (A'_2(x) \ln A'_2(x)) + (\text{Tr } A'(x)) (\ln \text{Tr } A'(x)). \quad (10)$$

Here, the  $\tilde{N} \times \tilde{N}$  matrix  $\tilde{A}$  is expressed in terms of the  $N \times N$  matrix  $A$  by Eq. (7), the integer  $\tilde{N} = n(\tilde{N}/n)$ , where the integer  $\tilde{N}/n = m$ , and the  $\tilde{N} \times \tilde{N}$  matrix  $A'(x) = \tilde{A} + x1_{\tilde{N}}$ , where  $1_{\tilde{N}}$  is the identity matrix in the  $\tilde{N}$ -dimensional space.

In an explicit form, inequality (10) reads

$$\begin{aligned} -\text{Tr} \left\{ \begin{pmatrix} a_{11} + x1_m & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} + x1_m & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} + x1_m \end{pmatrix} \ln \begin{pmatrix} a_{11} + x1_m & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} + x1_m & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} + x1_m \end{pmatrix} \right\} \\ -(\tilde{N}x + \text{Tr } A) \ln(\tilde{N}x + \text{Tr } A) \leq -\text{Tr} \left\{ \begin{pmatrix} \text{Tr}(a_{11} + x1_m) & \text{Tr } a_{12} & \cdots & \text{Tr } a_{1n} \\ \text{Tr } a_{21} & \text{Tr}(a_{22} + x1_m) & \cdots & \text{Tr } a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \text{Tr } a_{n1} & \text{Tr } a_{n2} & \cdots & \text{Tr}(a_{nn} + x1_m) \end{pmatrix} \right. \\ \left. \times \ln \begin{pmatrix} \text{Tr}(a_{11} + x1_m) & \text{Tr } a_{12} & \cdots & \text{Tr } a_{1n} \\ \text{Tr } a_{21} & \text{Tr}(a_{22} + x1_m) & \cdots & \text{Tr } a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \text{Tr } a_{n1} & \text{Tr } a_{n2} & \cdots & \text{Tr}(a_{nn} + x1_m) \end{pmatrix} \right\} \\ -\text{Tr} \{ (a_{11} + a_{22} + \cdots + a_{nn} + nx1_m) \ln(a_{11} + a_{22} + \cdots + a_{nn} + nx1_m) \}, \end{aligned} \quad (11)$$

where  $1_m$  is the identity matrix in the  $m$ -dimensional space.

For  $N = \tilde{N}$ ,  $\tilde{A} = A$ , and if  $x = 0$  and  $A \geq 0$ , Eq. (11) converts in Eq. (6), if  $\text{Tr } A = 1$ .

The matrix entropic inequality (11) is the main new relation found in our work. It can be applied to the “separable” matrix  $A$ , which has the form of convex sum  $A = \sum_k p_k A_k^{(1)} \otimes A_k^{(2)}$ , or to the entangled matrix  $A$ . If the matrix  $A$  is the diagonal one, inequality (11) is the inequality for real vectors and, if the components of the vectors are nonnegative, we have the entropic inequalities for the probability vectors.

### 3 Examples of $N = 3$ and 4

In this section, we present the inequalities for particular values of  $N$ . We consider the Hermitian  $4 \times 4$  matrix  $A$  given in the block form ( $4 = 2 \cdot 2$ ), i.e.,  $m = 2$  and  $n = 2$ ,  $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ , where

$$a_{11} = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}, \quad a_{12} = \begin{pmatrix} \rho_{13} & \rho_{14} \\ \rho_{23} & \rho_{24} \end{pmatrix}, \quad a_{21} = \begin{pmatrix} \rho_{31} & \rho_{32} \\ \rho_{41} & \rho_{42} \end{pmatrix}, \quad a_{22} = \begin{pmatrix} \rho_{33} & \rho_{34} \\ \rho_{43} & \rho_{44} \end{pmatrix}. \quad (12)$$

We have  $\text{Tr } a_{11} = \rho_{11} + \rho_{22}$ ,  $\text{Tr } a_{12} = \rho_{13} + \rho_{24}$ ,  $\text{Tr } a_{21} = \rho_{41} + \rho_{42}$ , and  $\text{Tr } a_{22} = \rho_{33} + \rho_{44}$ .

Let  $A = A^\dagger$ ,  $\text{Tr } A = \mu = \rho_{11} + \rho_{22} + \rho_{33} + \rho_{44}$ , and  $A \geq 0$ . In this case, we have the inequality

$$\begin{aligned}
& -\text{Tr} \left\{ \begin{pmatrix} \rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} \\ \rho_{21} & \rho_{22} & \rho_{23} & \rho_{24} \\ \rho_{31} & \rho_{32} & \rho_{33} & \rho_{34} \\ \rho_{41} & \rho_{42} & \rho_{43} & \rho_{44} \end{pmatrix} \ln \begin{pmatrix} \rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} \\ \rho_{21} & \rho_{22} & \rho_{23} & \rho_{24} \\ \rho_{31} & \rho_{32} & \rho_{33} & \rho_{34} \\ \rho_{41} & \rho_{42} & \rho_{43} & \rho_{44} \end{pmatrix} \right\} \\
& - (\rho_{11} + \rho_{22} + \rho_{33} + \rho_{44}) \ln(\rho_{11} + \rho_{22} + \rho_{33} + \rho_{44}) \\
& \leq -\text{Tr} \left\{ \begin{pmatrix} \rho_{11} + \rho_{22} & \rho_{13} + \rho_{24} \\ \rho_{31} + \rho_{42} & \rho_{33} + \rho_{44} \end{pmatrix} \ln \begin{pmatrix} \rho_{11} + \rho_{22} & \rho_{13} + \rho_{24} \\ \rho_{31} + \rho_{42} & \rho_{33} + \rho_{44} \end{pmatrix} \right\} \\
& - \text{Tr} \left\{ \begin{pmatrix} \rho_{11} + \rho_{33} & \rho_{12} + \rho_{34} \\ \rho_{21} + \rho_{43} & \rho_{22} + \rho_{44} \end{pmatrix} \ln \begin{pmatrix} \rho_{11} + \rho_{33} & \rho_{12} + \rho_{34} \\ \rho_{21} + \rho_{43} & \rho_{22} + \rho_{44} \end{pmatrix} \right\}. \tag{13}
\end{aligned}$$

If  $\text{Tr } A = \mu = 1$ , the matrix  $A$  can be interpreted either as the density matrix of the two-qubit state or as the density matrix of the qudit state with  $j = 3/2$ . The density matrix must satisfy inequality (13).

We can obtain an extra entropic inequality for the matrices created, in view of the portrait map applied to the matrix  $A$ . For this, we introduce the number  $\tilde{N} = N + 2 = 6$  and construct the  $6 \times 6$

matrix  $\tilde{A} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & 0 \end{pmatrix}$ . We assume  $\text{Tr } A = 1$ . Then, since  $6 = 2 \cdot 3$  and  $6 = 3 \cdot 2$ , we obtain the two following inequalities:

$$\begin{aligned}
-\text{Tr}(A \ln A) & \leq -\text{Tr} \left\{ \begin{pmatrix} \rho_{44} + \rho_{22} & \rho_{23} \\ \rho_{32} & \rho_{33} + \rho_{11} \end{pmatrix} \ln \begin{pmatrix} \rho_{44} + \rho_{22} & \rho_{23} \\ \rho_{32} & \rho_{33} + \rho_{11} \end{pmatrix} \right\} \\
& - \text{Tr} \left\{ \begin{pmatrix} \rho_{11} & \rho_{13} & 0 \\ \rho_{31} & \rho_{22} + \rho_{33} & \rho_{24} \\ 0 & \rho_{42} & \rho_{44} \end{pmatrix} \ln \begin{pmatrix} \rho_{11} & \rho_{13} & 0 \\ \rho_{31} & \rho_{22} + \rho_{33} & \rho_{24} \\ 0 & \rho_{42} & \rho_{44} \end{pmatrix} \right\}, \tag{14}
\end{aligned}$$

$$\begin{aligned}
-\text{Tr}(A \ln A) & \leq -\text{Tr} \left\{ \begin{pmatrix} \rho_{11} + \rho_{22} & \rho_{14} \\ \rho_{41} & \rho_{33} + \rho_{44} \end{pmatrix} \ln \begin{pmatrix} \rho_{11} + \rho_{22} & \rho_{14} \\ \rho_{41} & \rho_{33} + \rho_{44} \end{pmatrix} \right\} \\
& - \text{Tr} \left\{ \begin{pmatrix} \rho_{33} & \rho_{34} & 0 \\ \rho_{43} & \rho_{11} + \rho_{44} & \rho_{12} \\ 0 & \rho_{21} & \rho_{22} \end{pmatrix} \ln \begin{pmatrix} \rho_{33} & \rho_{34} & 0 \\ \rho_{43} & \rho_{11} + \rho_{44} & \rho_{12} \\ 0 & \rho_{21} & \rho_{22} \end{pmatrix} \right\}. \tag{15}
\end{aligned}$$

Thus, we showed that the matrix  $A$  (12) with  $\text{Tr } A = 1$  satisfies entropic inequalities (13)–(15).

In the case of two qubits, inequality (13) coincides with the quantum subadditivity condition, i.e., with the entropic inequality  $S(1, 2) \leq S(1) + S(2)$ , where the left-hand side of (13) is equal to the von Neumann entropy of the two-qubit state, and both terms in the right-hand side of (13) are the entropies of the first and second qubits, respectively. For qudit with  $j = 3/2$ , the inequality was discussed in [?, 21, 33].

For  $N = 3$ , we can choose  $\tilde{N} = N + 1 = 4$  and apply the obtained inequality to an arbitrary Hermitian

matrix  $A = \begin{pmatrix} \rho_{11} & \rho_{12} & \rho_{13} \\ \rho_{21} & \rho_{22} & \rho_{23} \\ \rho_{31} & \rho_{32} & \rho_{33} \end{pmatrix}$  and the matrix  $\tilde{A} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$  with blocks

$$a_{11} = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}, \quad a_{12} = \begin{pmatrix} \rho_{13} & 0 \\ \rho_{23} & 0 \end{pmatrix}, \quad a_{21} = \begin{pmatrix} \rho_{31} & \rho_{32} \\ 0 & 0 \end{pmatrix}, \quad a_{23} = \begin{pmatrix} \rho_{33} & 0 \\ 0 & 0 \end{pmatrix}.$$

We arrive at the inequality

$$\begin{aligned} & -\text{Tr} \left\{ \begin{pmatrix} \rho_{11} + x & \rho_{12} & \rho_{13} & 0 \\ \rho_{21} & \rho_{22} + x & \rho_{23} & 0 \\ \rho_{31} & \rho_{32} & \rho_{33} + x & 0 \\ 0 & 0 & 0 & x \end{pmatrix} \ln \begin{pmatrix} \rho_{11} + x & \rho_{12} & \rho_{13} & 0 \\ \rho_{21} & \rho_{22} + x & \rho_{23} & 0 \\ \rho_{31} & \rho_{32} & \rho_{33} + x & 0 \\ 0 & 0 & 0 & x \end{pmatrix} \right\} \\ & - (\rho_{11} + \rho_{22} + \rho_{33} + 4x) \ln(\rho_{11} + \rho_{22} + \rho_{33} + 4x) \\ & \leq -\text{Tr} \left\{ \begin{pmatrix} \rho_{11} + \rho_{22} + 2x & \rho_{13} \\ \rho_{31} & \rho_{33} + 2x \end{pmatrix} \ln \begin{pmatrix} \rho_{11} + \rho_{22} + 2x & \rho_{13} \\ \rho_{31} & \rho_{33} + 2x \end{pmatrix} \right\} \\ & - \text{Tr} \left\{ \begin{pmatrix} \rho_{11} + \rho_{33} + 2x & \rho_{12} \\ \rho_{21} & \rho_{22} + 2x \end{pmatrix} \ln \begin{pmatrix} \rho_{11} + \rho_{33} + 2x & \rho_{12} \\ \rho_{21} & \rho_{22} + 2x \end{pmatrix} \right\}, \end{aligned} \quad (16)$$

where  $x$  is such a number that  $A_0 + x \geq 0$  and  $A_0$  is the minimum negative eigenvalue of the Hermitian matrix  $A$ . If the  $3 \times 3$  matrix  $A$  is the density matrix of a qutrit state (we assume  $x = 0$  and  $\text{Tr } A = 1$ ), inequality (16) coincides with the subadditivity condition considered in [19].

## 4 Conclusions

To conclude, we point out the main results of our work.

We found the new matrix inequality valid for an arbitrary Hermitian matrix; it is given by Eq. (11). We introduced the notion of matrix information; see Eq. (5).

For arbitrary  $N \times N$  matrix  $A$ , we constructed the matrix portrait  $\Phi(A)$ , which is the linear map  $A \rightarrow A' = \Phi(A)$ , being an analog of the partial tracing procedure used to obtain the matrix factors  $B$  and  $C$  presenting the matrix  $A$  in the form of tensor product of these factors,  $A = B \otimes C$ . Employing the matrix portraits  $B$  and  $C$  and embedding the matrices  $A$ ,  $B$ , and  $C$  in the linear space of higher dimensions, we obtained new entropic matrix inequalities written for the Hermitian matrix  $A$  in the explicit form. Considering the subset of the set of matrices  $A$ , which contains all density  $N \times N$  matrices of the systems of qudits, we derived new entropic matrix and information inequalities for the density matrices.

Due to the procedure suggested here, we extended the known entropic subadditivity condition for bipartite quantum systems to the case of arbitrary single qudit systems. The method to obtain for a single qudit state all entropic inequalities known for composite systems, including the inequalities for the von Neumann entropy and  $q$ -entropy [34–36] along with the Bell-like inequalities [28–30], can be formulated as a straightforward continuation of the tools demonstrated in this work.

We presented the map of the  $N \times N$  matrix  $A \rightarrow \Phi(A)$  in the case of factorization  $N = nm$ . Repeating the map algorithm step-by-step, one can construct a chain of maps for  $N = \prod_{k=1}^M N_k$ , where  $N_k$  are integers, and also in the case where  $\tilde{N} = N + s = \prod_{k=1}^{\tilde{M}} N_k$ .

Since we understood that a single qudit state could have the entanglement properties analogous to the entanglement properties of multiqubit systems, we suggested to apply this knowledge to study the resource of entanglement to be used for quantum computing, analogously as it takes place in the case of composite quantum systems [31].

The obtained map of the density matrix of a single qudit state on the density matrix of a multiqubit state, including the  $N$ -qubit state, provides the possibility to classify the quantum channels transforming the separable states into entangled states, and vice versa, of the single qudit. This possibility is related to the identity of the  $N$ -dimensional Hilbert-space properties, which do not depend on the interpretation of the Hilbert space as the space of states of composite or noncomposite systems. Since there exists the strong subadditivity condition for the density matrix of the three-partite system [23], we can obtain a new matrix inequality, which is an analog of this condition, for an arbitrary Hermitian  $N \times N$  matrix, including the density matrix of the single qudit state. We continue the consideration of the found matrix inequalities in the form of relations for qudit tomograms of classical and quantum system states [37–40], employing the inequalities for the probability vectors depending on the parameters of the unitary matrix in a future publication.

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